Dynamic Bond-Stock-Commodity Portfolio Optimization: A Numerical Example

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This research aims at examining an explicit investment policy of mixed bond-stock-commodity dynamic portfolio problems under a simple interest rate model and mean-reverting commodity prices with estimated parameters. The policy recommends a relatively negative relationship between risk-aversion factor and riskier assets, stocks and commodities, while it proposes a positive relationship to that of zero-coupon bonds. This is consistent with the professional advice that investors who can tolerate more risk should invest more in riskier assets such as stocks and commodities. The paper also finds a positive relationship between bond investment and time horizon. A relationship between investment in commodity and time horizon, however, is more complicated; investors reduce their portions with longer investment horizon but will finally increase allocation in the long run. Finally, the study of correlation and financial allocation results in various outcomes depending on estimated parameters. In summary, allocation in each asset will be increased when relevant correlations convert to -1.

Keywords: Dynamic Asset Allocation; Risk Aversion; Mean Reversion; Commodities

INTRODUCTION

Commodities have been emerging as an increasingly important class of assets and are claimed to have value-added effectiveness due to their diversification benefits. Ibbotson Associates (2006), exhibiting the correlation coefficients of annual total returns (1970-2004), provides intuitive evidence of the low correlation of commodities with traditional asset classes. Of the seven asset classes, treasury bills and commodities are the only two asset classes with negative average correlations to the other asset classes. In addition, commodities are positively correlated with inflation (see, for example, Erb and Harvey, 2006; Gorton and Rouwenhorst, 2006). There are some studies concerned in the diversification and inflation hedge effects of commodities, especially research of short-term investment using the framework of Markowitz (1952). Despite the rise of commodity investment, portfolios of most investors are still comprised mainly of traditional assets as stocks and bonds, so it is important to keep them in investment decisions. In fact, given the growing importance of commodities, it would be relevant to set up a portfolio consisting of bonds, stocks, and commodities. To accomplish such an objective, this study combines characteristics from models including term structure of interest rates, models of mean-reversion, and models including commodity as an alternative asset.

To introduce bonds as one of the investment choices, it is necessary to include the model of term structure of interest rate dynamics into portfolio problems. Sørensen (1999), Brennan and Xia (2000), and Korn and Kraft (2001) consider investment problems under stochastic interest rates of the Vasicek (1977) type where an investor with CRRA utility can invest in a bank account, stocks, and bonds. They argue that the optimal investment strategy is a simple combination of a speculative term and a hedge term. While the former explains the need to optimize an immediate risk-return profile in a mean-variance framework, the latter describes how the investor protects stochastic behaviors of interest rates. Particularly, Brennan and Xia (2000) show that, consistent with popular recommendations, the bond-stock ratio has a positive relationship with the degree of risk-aversion. Using more advanced interest rate model assumptions, Munk and Sørensen (2004) investigate investment strategies similar to those assuming simple interest rate models. However, one of their key results is that the hedge portfolio is more sensitive to the current form of the term structure than to the specific dynamics of interest rates. Thus, not only is it easier to apply basic models such as the Vasicek model in our research, but it is also possibly sufficient in terms of correctness and practicality. Another type of allocation problems related to this paper is the optimal portfolio with the stochastic market price of risk. As some empirical studies suggest evidence of mean reversion in stock returns, Kim and Omberg (1996), Wachter (2002), and Munk, Sørensen, and Vinther (2004) achieve exact optimal investment strategies in a set-up where the market price of risk is identical to the Sharpe ratio of the stock. Instead of assuming mean reversion in stock prices, we will apply the stochastic characteristics of market price of risk into commodity prices as in Schwartz (1997).

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As mentioned earlier about the importance and popularity of commodities, most existing studies on commodity investment apply the one-period mean-variance optimization framework of Markowitz (1952). However, much less research efforts have been devoted to long-term allocation strategies. The closest literature to the present paper is the work by Dai (2009), studying dynamic asset allocation using the Martingale approach with a focus on introducing commodities into portfolio management. The main difference between Dai (2009) and our work is that Dai's portfolio includes stocks and commodities while this study includes bonds, stocks, and commodities, thus adding more complexity in the search for an optimal investment strategy. Our previous study (Maneenop, 2012) finds an explicit investment strategy with hedge variations in interest rates and the commodity market price of risk. Assuming that the spot commodity price has a significant negative effect on the risk premium as in Schwartz (1997) and Dai (2009), no conclusions can be made regarding to the direction of zero-coupon bond investment resulting from a rise in commodity prices. However, it may be concluded that there are inverse relationships between commodity prices and positions in stocks and commodities in the portfolio.

To complement mathematical results and provide more details in inconclusive parts of our previous research, the present paper aims at showing an illustrative example based on such a strategy of dynamic bond-stock-commodity allocation under a stochastic opportunity set of investors holding constant relative risk-aversion (CRRA) utility. It also investigates whether two traditional pieces of advices that (1) aggressive investors should hold stocks while conservative investors should hold bonds; and (2) long-term investors invest more in stocks than short-term investors are true. This study will discuss whether those two beliefs hold up well when approached academically and how introduction of commodities affects in these rules of thumb. Moreover, the paper considers the relationship between movement in commodity prices and changes in investment strategies using an illustration. In short, we discuss whether an increase in commodity prices leads to long or short positions in other assets, especially, a relationship between zero-coupon bond investment and a rise in commodity prices. To provide more contribution to the field, investment horizon and correlation effect to investment strategies will also be discussed in this article. In spite of delivering clearer pictures compared to the originally mathematical work in Maneenop (2012), there are at least two limitations in this study. Estimated parameters used in this research are collected from previously relevant literature instead of estimating by ourselves. Also, findings in the last part of this present article about correlation effect need further studies to support some results.

In the next section, we set up investment asset dynamics and examine an optimal asset allocation strategy. Then, we will analyze such an optimal solution with illustrative results. Finally, we conclude the paper in the last section.

INVESTMENT ASSET DYNAMICS AND OPTIMAL ASSET ALLOCATION

In this section we introduce the investment asset dynamics and the optimal asset allocation. The interest rate dynamics are explained by Vasicek (1977) process.

$$\mathrm{d}r_t = \kappa(\bar{r} - r_t)\mathrm{d}t - \sigma_r \mathrm{d}Z_{rt} \tag{1}$$

where $\bar{r}$ indicates the long-run mean of the interest rate, κ denotes the degree of mean reversion, σ_r is the volatility of the interest rate, and Z_{rt} is a standard Brownian motion. Such a process leads to a zero-coupon bond dynamics with maturity $\bar{T}$ given by

$$\mathrm{d}B_t^{\bar{T}} = B_t^{\bar{T}}\left[\left(r_t + \sigma_B(r_t,t)\lambda_1\right)\mathrm{d}t + \sigma_B(r_t,t)\mathrm{d}Z_{rt}\right] \tag{2}$$

where

$$\sigma_B(r_t,t) = \sigma_r b\left(\bar{T}-t\right),\ b\left(\bar{T}-t\right) = \frac{1}{\kappa}\left(1-\mathrm{e}^{-\kappa\left(\bar{T}-t\right)}\right)$$

with the constant parameter, λ_1, is the premium on interest rate risk.

The dynamics of a stock price, S_t, are assumed as the following stochastic differential equation

$$\mathrm{d}S_t = S_t\left[\left(r_t + \sigma_s\psi\right)dt + \rho_{rs}\sigma_s\mathrm{d}Z_{rt} + \sqrt{1-\rho_{rs}^2}\sigma_s\mathrm{d}Z_{st}\right] \tag{3}$$

where σ_s describes the stock volatility, ψ denotes the stock market price of risk, and the product of these two parameters expresses expected excess return from equity investment. The correlation between returns in stocks and interest rates is denoted by ρ_{rs}. Z_{st} is another standard Brownian motion and independent of Z_{rt}.

The spot commodity price, C_t, is assumed to follow the one-factor model in Schwartz (1997) given by

$$\mathrm{d}C_t = \theta\left(\mu_c - \ln C_t\right)C_t\mathrm{d}t + \sigma_c C_t \mathrm{d}\hat{Z}_{ct} \tag{4}$$

where μ_c is the long–run mean of the spot price, θ denotes the degree of mean reversion, σ_c explains the commodity volatility, and $\hat{Z}_{ct}$ is a standard Brownian motion. The model is popular for modeling energy and agricultural commodities and aims at introducing mean reversion to the long-run mean, μ_c.

Under risk-neutral measure, it is possible to find the futures price of such a commodity with the following market price of risk

$$\lambda_{ct} = \lambda_{c1} + \lambda_{c2}X_t,\ \ X_t = \ln C_t \tag{5}$$

As discussed in Dai (2009), the above equation is inspired by empirical evidence that expected returns are time-varying and can be predicted by some instrumental variables such as spot commodity prices themselves.

Also, following Dai (2009), the commodity price process may be adapted to the following process

$$\mathrm{d}V_t = V_t\left[\left(r_t + \sigma_c\lambda_{ct}\right)\mathrm{d}t + \sigma_c\mathrm{d}\hat{Z}_{ct}\right] \tag{6}$$

where V_t is the self-financing portfolio characterized as a new asset class of commodities.

With this process, it is possible to set up the dynamics of three assets, bonds, stocks, and commodities consistent with asset allocation problems of Merton (1969, 1971). The dynamics of three assets can be expressed in terms of the following matrix

$$\mathrm{d}\underline{P}_t = \begin{bmatrix} \mathrm{d}B_t^{\bar{T}} \\ \mathrm{d}S_t \\ \mathrm{d}V_t \end{bmatrix} = \mathrm{diag}(P_t)\left[\left(r\underline{1} + \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right)\underline{\lambda}(\lambda_{3t})\right)\mathrm{d}t + \underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right)\mathrm{d}\underline{Z}_t\right] \tag{7}$$

where

$$\underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right) = \begin{bmatrix} \sigma_B(r_t, t) & 0 & 0 \\ \rho_{rs} & \bar{\rho}\sigma_s & 0 \\ \rho_{cr} & \rho_{sc} & \tilde{\rho}\sigma_s \end{bmatrix},\ \ \underline{\lambda}(\lambda_{3t}) = \left[\lambda_1 \quad \lambda_2 \quad \lambda_{3t}\right]^{\mathrm{T}},\ \ \bar{\rho} = \sqrt{1-\rho_{rs}^2},\ \ \tilde{\rho} = \sqrt{1-\rho_{cr}^2-\rho_{sc}^2}$$

and where parameters with one underline are vectors and parameters with two underlines denotes matrices. The vector of market price of risk is composed of two constants, λ_1 and λ_2, with respect to Z_{rt} and Z_{st} respectively; and another state variable, λ_{3t}, with respect to Z_{ct}. Note that the Brownian motion Z_{ct} is independent of Z_{rt} and Z_{st}. The parameter λ_{3t} is stochastic and dependent on the commodity price. λ_2 and λ_{3t} can be inferred from the above price dynamics as

$$\lambda_2 = \left(\psi - \rho_{\mathrm{rs}}\lambda_1\right)/\bar{\rho} \tag{8}$$

and

$$\lambda_{3t} = \left(\lambda_{ct} - \rho_{cr}\lambda_1 - \rho_{sc}\lambda_2\right)/\tilde{\rho} \tag{9}$$

As λ_{ct} depends on the commodity price, The market price of risk, λ_{3t}, can be expressed in the following dynamics

$$\mathrm{d}\lambda_{3t} = \theta\left(\bar{\lambda}_{3t} - \lambda_{3t}\right)\mathrm{d}t + \left(\lambda_2\sigma_c/\tilde{\rho}\right)\left[\rho_{cr} \quad \rho_{sc} \quad \tilde{\rho}\right]\mathrm{d}\underline{Z}_t \tag{10}$$

where $\bar{\lambda}_{3t} = \left(1/\tilde{\rho}\right)\left(\lambda_{c2}\alpha + \lambda_{c1} - \rho_{cr}\lambda_1 - \rho_{sc}\lambda_2\right)$

The stochastic characteristics of the commodity market price of risk and the interest rate as mentioned earlier generates a stochastic investment opportunity set effect in hedging terms. The optimal investment strategy will be different from the case with static mean-variance framework. Two state variables, r_t and λ_{3t}, can be written in the following matrix

$$\begin{bmatrix} \mathrm{d}r_t \\ \mathrm{d}\lambda_{3t} \end{bmatrix} = \begin{bmatrix} \kappa\left(\bar{r} - r_t\right) \\ \theta\left(\bar{\lambda}_{3t} - \lambda_{3t}\right) \end{bmatrix}\mathrm{d}t + \begin{bmatrix} -\sigma_r & 0 & 0 \\ \dfrac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\rho_{cr} & \dfrac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\rho_{sc} & \lambda_{c2}\sigma_c \end{bmatrix}\mathrm{d}\underline{Z}_t \tag{11}$$

Following Munk (2010), the wealth process, W_t, can be expressed as

$$\mathrm{d}W_t = W_t\left[r_t + \underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right)\underline{\lambda}(\lambda_{3t})\right]\mathrm{d}t + W\underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}\left(r_t, \lambda_{3t}, t\right)\mathrm{d}\underline{Z}_t \tag{12}$$

where $\underline{\pi}_t$ is the three-dimensional vector of investment portion at time t in the portfolio consisted of bonds, stocks, and commodities. The remains of this wealth, $1 - \pi_B - \pi_S - \pi_V$, is invested in the risk-free asset.

An investor is assumed to maximize utility from the terminal wealth, W_T, with respect to a power utility function. The indirect utility function is given by

$$J\left(W, r, \lambda_{3t}, t\right) = \sup E_{W,r,\lambda_{3t},t}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \tag{13}$$

where the parameter $\gamma > 0$ describes the risk tolerance level of an investor. Using the dynamic programming approach or the Martingale approach, the optimal asset allocation of a power utility investor is stated in the following theorem.

Theorem 1: *The indirect utility of wealth function of a CRRA investor is given by*

$$J\left(W, r, \lambda_{3t}, t\right) = \frac{1}{1-\gamma}\left(W\exp\left\{A_0(T-t) + A_1(T-t)r + A_2(T-t)\lambda_{3t} + A_3(T-t)\lambda_{3t}^2\right\}\right)^{1-\gamma} \tag{14}$$

where

$$\begin{aligned} A_0(\tau) = {} & \frac{1}{2\gamma}\left(\lambda_1^2 + \lambda_2^2\right)\tau + \left(\kappa\bar{r} + \frac{\gamma-1}{\gamma}\sigma_r\lambda_1\right)\int_0^\tau A_1(s)\mathrm{d}s + \left(\theta\bar{\lambda}_{3t} - \frac{\gamma-1}{\gamma}g\right)\int_0^\tau A_2(s)\mathrm{d}s + \frac{1}{2}\sigma_*^2\int_0^\tau A_3(s)\mathrm{d}s \\ & -\frac{\gamma-1}{2\gamma}\sigma_r^2\int_0^\tau A_1^2(s)\mathrm{d}s + \frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\sigma_r\rho_{cr}\int_0^\tau A_1(s)A_2(s)\mathrm{d}s + \sigma_*^2\int_0^\tau A_2^2(s)\mathrm{d}s \end{aligned}$$

$$A_1(\tau) = \frac{1}{\kappa}\left(1 - \mathrm{e}^{-\kappa\tau}\right),\ A_2(\tau) = \left[\frac{2af\left(\mathrm{e}^{d\tau/2} - 1\right)^2}{d\left(\mathrm{e}^{d\tau} - 1\right)}\right]A_3(\tau),\ A_3(\tau) = \frac{2a\left(e^{d\tau} - 1\right)}{\left(b+d\right)\left(e^{d\tau} - 1\right) + 2d},$$

with

$$a = \frac{1}{\gamma},\ b = 2\left(\theta + \frac{\gamma-1}{\gamma}\lambda_{c2}\sigma_c\right),\ c = -\frac{\gamma-1}{\gamma}\sigma_*^2,\ d = \sqrt{b^2 - 4ac},\ \sigma_*^2 = \left(\frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\rho_{cr}\right)^2 + \left(\frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\rho_{sc}\right)^2 + \left(\lambda_{c2}\sigma_c\right)^2,$$

$$f = \theta\bar{\lambda}_{3t} - \frac{\gamma-1}{\gamma}\left(g + \frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\sigma_r\rho_{cr}A_1(\tau)\right),\ g = \frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\left(\rho_{cr}\lambda_1 + \rho_{sc}\lambda_2\right)$$

The optimal risk asset allocations at time t is given by

$$\pi_B = \frac{1}{\gamma\sigma_B(r,t)}\left[\lambda_1 - \frac{\lambda_2\rho_{rs}}{\bar{\rho}} - \frac{\lambda_{3t}}{\bar{\rho}\tilde{\rho}}\left(\bar{\rho}\rho_{cr} - \rho_{rs}\rho_{sc}\right)\right] + \frac{\gamma-1}{\gamma}\frac{\sigma_r A_1(\tau)}{\sigma_B(r,t)} \tag{15}$$

$$\pi_S = \frac{1}{\gamma\sigma_S}\left(\frac{\lambda_2}{\bar{\rho}} - \frac{\lambda_{3t}\rho_{sc}}{\bar{\rho}\tilde{\rho}}\right) \tag{16}$$

$$\pi_V = \frac{\lambda_{3t}}{\gamma\tilde{\rho}\sigma_C} - \frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\left[A_2(\tau) + A_3(\tau)\lambda_{3t}\right] \tag{17}$$

The proof of the theorem along with mathematical discussions can be found in Maneenop (2012). In summary, allocation in zero-coupon bond and commodity is a combination of a speculative term and a hedge term whereas allocation in stocks consists only of a speculative term. Hedge terms in the zero-coupon bond and the commodity explain protection against change in interest rates and change in commodity market price of risk, respectively. Next section will discuss the theorem in more details using estimated parameters.

A NUMERICAL EXAMPLE

This section illustrates the strategy in Theorem 1 with historical estimates of mean returns, standard deviations, and correlations as representative of future investment opportunities. Estimates are taken from Munk (2010), Dai (2009), and Ibbotson Associates (2006). The average real return on the U.S. stock market is 8.70% with a standard deviation of σ_S = 20.20%, while the average real return on bonds is 2.10% with a standard deviation of 10.00%. The average real U.S. short-term interest rate is $\bar{r}$ = 1.00%. and the correlation between stock returns and bond returns is ρ_{rs} = 0.20. The average long-run mean of the commodity price, μ_c, is 5.02, the degree of mean-reversion, θ, is 0.12, with a standard deviation of σ_c = 19.90%. Market prices of risk, λ_{c1} and λ_{c2} equal 3.32 and -0.67. Correlation coefficients, ρ_{rs}, ρ_{sc}, and ρ_{cr} are 0.20, -0.10, and -0.30, respectively. Finally, the initial spot commodity price is set at 130, the same as in Dai (2009).

Hedge and total portfolio with risk aversion

Figure 1 presents the relation between asset allocation and risk aversion with 5-year and 20-year investment time horizons, respectively. As can be seen, investment in bonds and risk aversion factor has a positive relationship. This implies that more conservative investors take larger positions in zero-coupon bonds. For stock investment, it is also obvious that conservative investors allocate less in stocks compared to aggressive investors. These two implications are the same as the professional advice that investors who can tolerate less risk should invest more in safer assets such as the zero-coupon bond. However, we notice that allocation to the stock is independent from the investment horizon. This contradicts with traditional advice that the stock weight should increase with investment horizon.

The relation between risk aversion factor and commodity investment is indeterminate for the investment in hedge portfolio. With a time horizon of 5 years, commodity allocation decreases as investors are more conservative. But when the risk-aversion factor is greater than 2, investment in commodities seems to rise. The case with 20-year time horizon also provides a similar phenomenon but with less inconsistency. Such a surprising increase in commodity investment is due to an increase in the hedge term of commodity investment. This result also leads to a difference between investment in stocks and commodities.

Bond/stock ratio and bond/commodity ratio

Figure 2 shows the relation between investment quotients and risk aversion. For each investment horizon, bond/stock and bond/commodity ratios increases with increasing risk aversion. This is the same as previous studies e.g. Sørensen (1999), Brennan and Xia (2000). Moreover, for investors with risk aversion less than one, the ratios decrease with increasing time horizon. These two results, therefore, support the aforementioned rules of thumb except for the case of risk aversion higher than one for the second result.

Investors with low risk aversion invest more in stocks and commodities than highly risk-averse investors since stocks and commodities are basically riskier than bonds. Correspondingly, longer-term investors invest more in riskier assets compared to shorter-term investors as stock and commodity volatilities are likely to reduce in the long run. However, for risk aversion factors greater than one, the ratios increase with increasing time horizon, contradicting the professional advice. This conflict is probably a consequence of the stronger risk aversion effect compared with the time horizon effect.

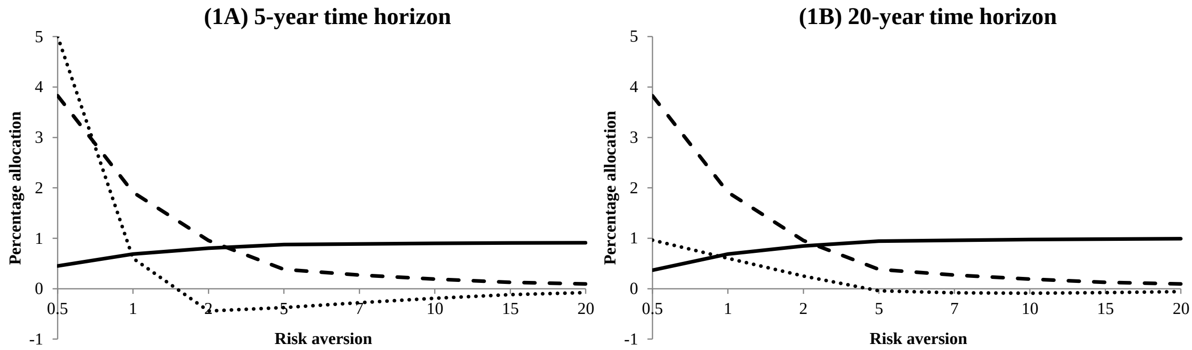

Figure 1. Relation between asset allocation risk aversion. The solid lines correspond to allocation in zero-coupon bonds, the dashed lines correspond to allocation in stocks, and the dotted lines correspond to allocation in commodities. The spot commodity price is assumed to be 130, close to that of the median value in Dai (2009).

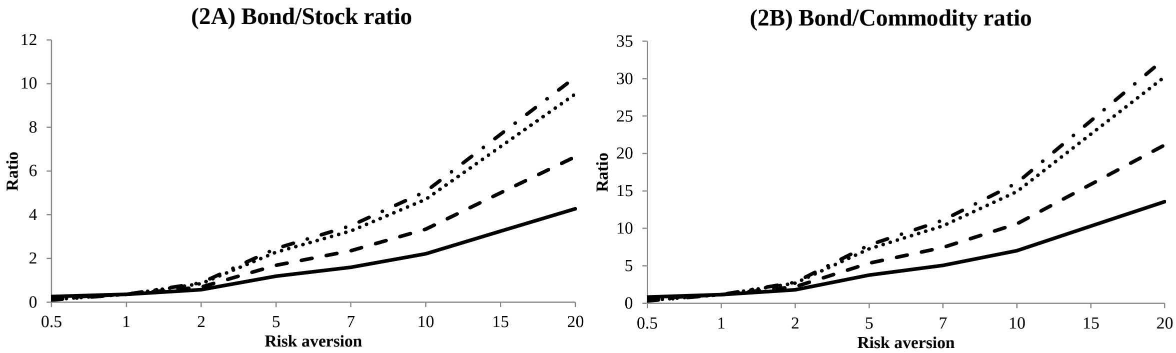

Figure 2. Relation between relative investments and risk aversion. The solid lines correspond to an investor with 1-year horizon, the dashed lines correspond to an investor with 2-year horizon, the dotted lines correspond to an investor with 5-year horizon, and the dot-dashed lines correspond to an investor with 10-year horizon. The spot price is assumed to be 130.

Horizon effect

As shown in Figure 3A, with $\gamma > 1$, bond investment has a positive relationship with the time horizon. It is likely that asset volatilities will reduce in the long-run so that investors can increase their portions in risky assets when considering long term investment.

By differentiating the commodity wealth fraction with respect to time horizon, we obtain the following derivative

$$\frac{\partial \pi_V}{\partial \tau} = -\frac{\gamma - 1}{\gamma} \frac{\lambda_{c2}}{\tilde{\rho}} \left[A_2'(\tau) + A_3'(\tau)\lambda_{3t} \right] \tag{18}$$

It can be shown that $A_3'(\tau)$ is always positive while $A_2'(\tau)$ can be either positive or negative depending on parameters. Thus, it is inconclusive whether the investment horizon has positive or negative effect in commodity investment. However, if λ_{3t} is high enough (which means lower initial spot price), the derivative will be positive and lead to a higher commodity allocation for a longer term investor. We can see from Figure 3B that, for short-term investments, investments decrease for longer investment horizons. However, for long-term investments, investments increase for longer investment horizons. Additionally, conservative investors slowly increase their portions in commodity allocation compared to aggressive investors. This is possibly due to fear of investment adaptation of conventional investors. Later, we will discuss more about horizon effect to commodity allocation.

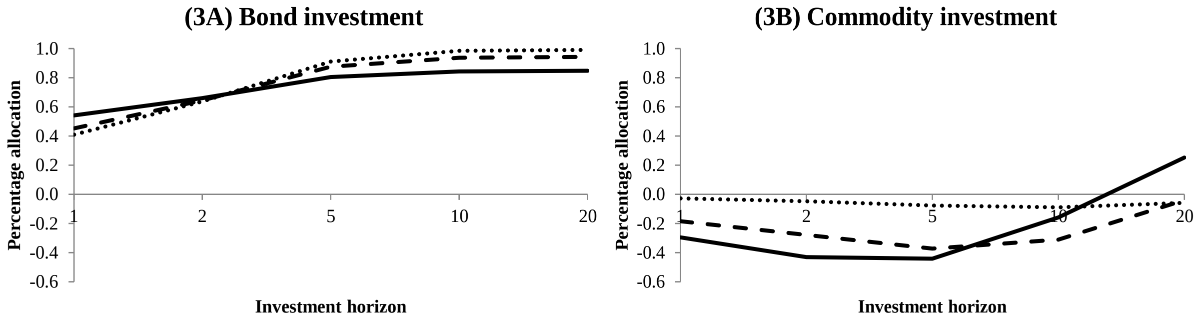

Figure 3. Relation between allocation in each asset and investment horizon. The solid lines correspond to an investor with γ = 2, the dashed lines correspond to an investor with γ = 5, and the dotted lines correspond to an investor with γ = 20. The spot price is assumed to be 130.

Change in spot initial price

As studied in Maneenop (2012), the relationship between change in spot initial price of commodities and investment in zero-coupon bonds is unclear when considering only mathematical models. The term $-\lambda_{3t}\left(\bar{\rho}\rho_{cr}-\rho_{rs}\rho_{sc}\right)/\bar{\rho}\tilde{\rho}$ could be either positive or negative depending on correlation factors, ρ_{cr}, ρ_{rs}, and ρ_{sc}. Regularly, empirical studies (e.g. Ibbotson Associates, 2006) suggest positive ρ_{rs} and negative in ρ_{sc} and ρ_{cr}. Also, Dai's (2009) finding that λ_{c2} is significantly less than zero, along with the empirical test of Schwartz (1997) concluding that the spot commodity price has significantly negative effect on the risk premium, suggests an inverse relationship between the commodity price, C_t, and the market price of risk, λ_{3t}. Plugging relevant parameters into the model, Figure 4A suggests a negative relationship between an increase in initial commodity prices and zero-coupon bond investments. This is mainly due to the negative correlation between commodity returns and bond returns.

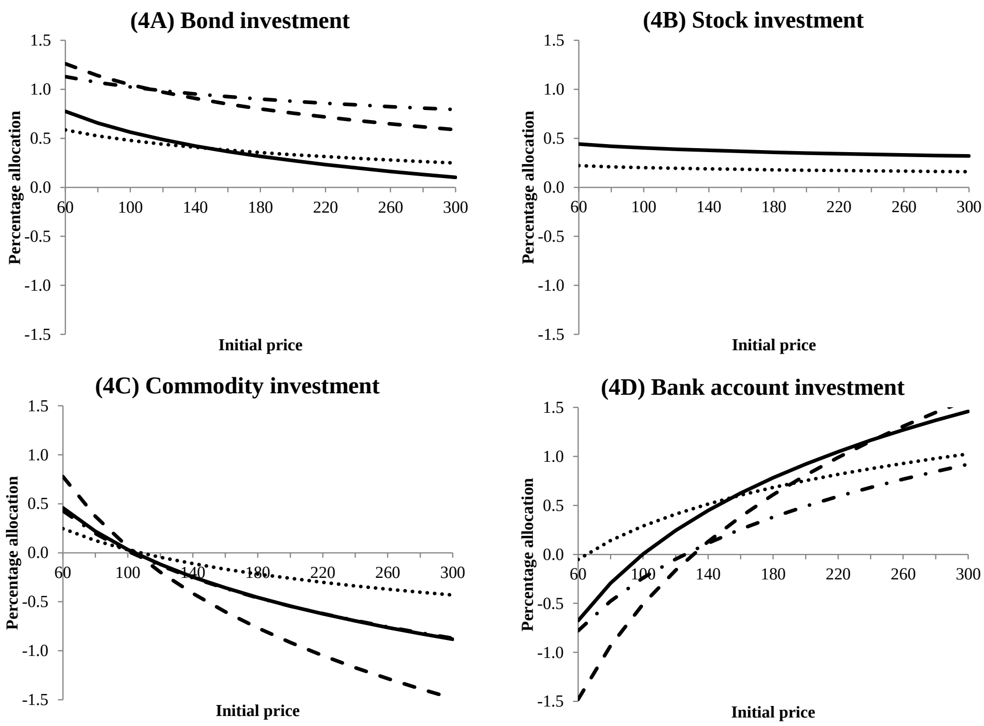

Figure 4. Relation between allocation in each asset and initial commodity price. The solid lines correspond to an investor with γ = 5 and 1-year investment horizon, the dashed lines correspond to an investor with γ = 5 and 10-year investment horizon, the dotted lines correspond to an investor with γ = 10 and 1-year investment horizon, and the dot-dashed lines correspond to an investor with γ = 5 and 10-year investment horizon. As stock allocation does not depend on time horizon, Figure 4B depicts only two different allocations.

Stock allocation depends only in the speculative term as there are no risks to hedge as in bonds and commodities. By examining (17), it is clear from the equation that if ρ_{sc} is less than zero, the investment in stock has negative relationship with the commodity price. It may be interpreted that when the commodity price falls, this indicates suggests an increase in the stock price; and investors, therefore, should invest in stock. Figure 4B confirms such a result but with gradual change in stock allocation.

For the commodity itself, the result is similar to Munk et al. (2004) and Dai (2009) that the hedge term includes both $A_2(\tau)$ and $A_3(\tau)$. Focusing only on the first term, the speculative portfolio, it can be implied that, with λ_{c2} less than zero, there is a negative relationship between the commodity price and the speculative term. Next, we concentrate on the hedge term and recall the value of $A_3(\tau)$. The fact that d is greater than b in Theorem 1 indicates the positivity of this term. This, therefore, leads to an inverse relationship between the commodity price and the hedge portfolio. As can be seen in Figure 4C, an increase in spot prices leads to reduction in commodity investment. In addition, as discussed in the previous subsection about horizon effect to commodity allocation, the figure here suggests that low spot prices lead to greater allocation to commodities. It is certain to settle that the investor should invest in commodities when the commodity price decreases and reduce the portion when the commodity price increases.

In brief, an increase in commodity prices tends to decrease investors' positions in risky assets. Moreover, conservative investors smoothly reduce their positions compared to aggressive investors. One of the possible explanations is that conventional investors adapt investment behavior slower than aggressive investors. As portions in all risky assets reduce, position in bank account must increase to balance the total portfolio as shown in Figure 4D.

Correlation effect

Figure 5 shows relationship between commodity investment and selected correlations with gamma of 5 and time horizon of 10 years. Figure 5A shows the relationship between bond investment and two correlations related to bond investment. Not surprisingly, if both correlations convert to -1, investors tend to increase their portions in zero-coupon bonds. On the contrary, if both correlations go to 1, investors reduce their positions in bonds. Investors use zero-coupon bonds as diversifying tools to maximize their profits. However, the effect of correlation between stock returns and bond returns is greater than that of the correlation between bond returns and commodity returns. This is due to the bond allocation feature that ρ_{rs} affects the investment via both λ_{c2} and λ_{3t} whereas ρ_{cr} only affects λ_{3t}.

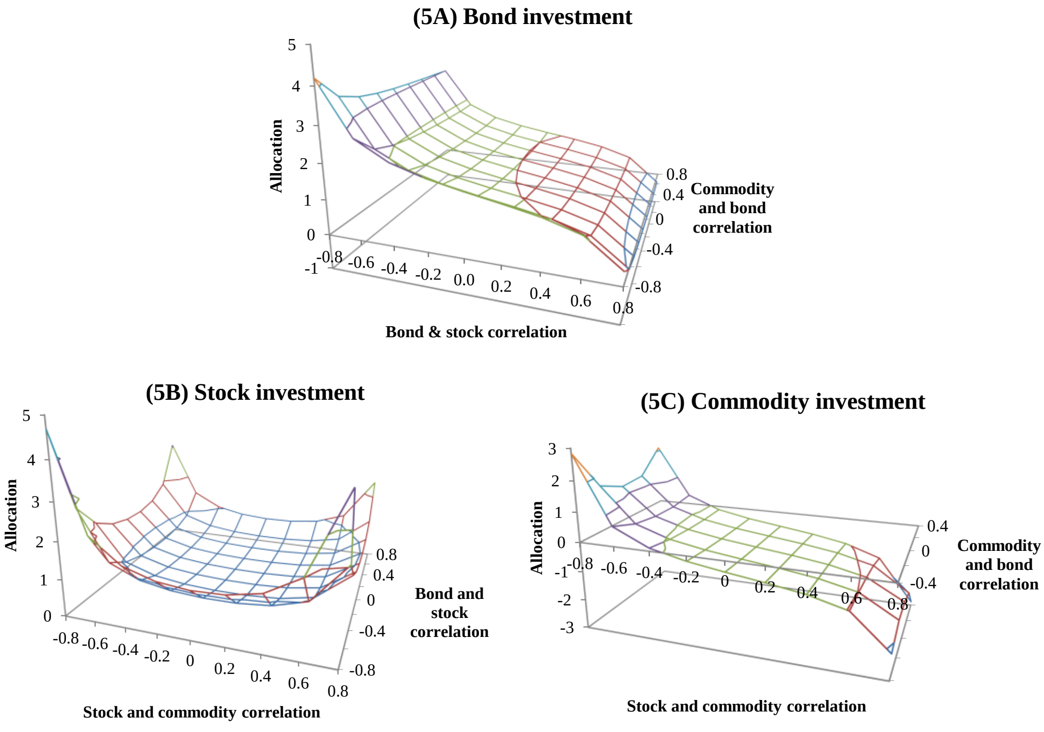

Figure 5. Relation between allocation in each asset and relevant correlations with gamma of 5 and time horizon of 10 years.

As can be seen in Figure 5B, a correlation between stock returns and commodity returns reduces with increasing investment in commodity. This is the diversification benefit of commodities. Specifically, with such a correlation closer to -1, and a correlation between commodity returns and bond returns divergent from zero, investors tend to increase their investment in commodities. The negativity of λ_{c2} plays an important role to the result of a negative relationship between stock-commodity correlation and commodity allocation. If λ_{c2} is positive, the shape of this figure will transform to a convex cone shape (the result is not shown here).

For stock investment, surprisingly, when absolute values of correlations relevant to stocks convert to 1, investors increase their positions in stock allocation. In the opposite case, when absolute values are close to zero, they reduce their portions in stocks. It is easily understandable that investors gain most from diversification effect when both correlations convert to -1 but it is questionable otherwise. Therefore, further studies on correlation effect to allocation in each asset are highly suggested.

CONCLUSIONS

This study has aimed at examining the bond-stock-commodity portfolio in the framework of dynamic asset allocation as in Merton (1969). The commodity market price of risk is stochastic and dependent on a spot commodity price, while the spot price itself has the property of mean reversion. The paper also assumes the simple Vasicek interest rate. Using closed-form solutions in Maneenop (2012), the study illustrates an optimal investment strategy with estimated parameters for a portfolio including zero-coupon bonds, stocks, and commodities.

The allocation to zero-coupon bonds and commodities were made for speculative or myopic purposes, as well as for intertemporal hedging purposes whereas the optimal allocation to the stock is solely speculative and dependent on the spot commodity price. Positions in stocks and commodities, the riskier assets, have negative relationships with investor’s risk tolerance, while positions in zero-coupon bonds, the less risky assets, have the opposite result. This is consistent with the professional advice that investors who can accept more risk should invest more in riskier assets.

Regarding to horizon effect, bond investment has a positive relationship with the time horizon. It is likely that asset volatilities will reduce in the long-run so that investors can increase their portions in risky assets when considering long term investment. However, for commodity investment, investors tend to decrease their portions as investment horizon is longer but will eventually increase allocation in the long run. Besides, conventional investors slowly increase their portions in commodity allocation compared to aggressive investors. This is possibly due to fear of investment adaptation of conventional investors.

Assuming the spot commodity price has a significant negative effect on the risk premium as in Schwartz (1997) and Dai (2009), all positions in risky assets decrease with increasing commodity prices. Moreover, conservative investors smoothly reduce their positions compared to aggressive investors. This result is a complement to our previously relevant article. One of the possible explanations is that conservative investors adapt investment behavior slower than aggressive investors. As portions in all risky assets reduce, position in bank account, thus, increase to balance the total portfolio.

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