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“Dynamic Bond-Stock-Commodity Portfolio Optimization: A Numerical Example”

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มหาวิทยาลัยธรรมศาสตร์ ท่าพระจันทร์

DYNAMIC BOND-STOCK-COMMODITY PORTFOLIO OPTIMIZATION: A NUMERICAL EXAMPLE

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Faculty of Commerce and Accountancy
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การประชุมวิชาการ “*ศาสตราจารย์สังเวียน อินทรวิชัย* ด้านตลาดการเงินไทย”
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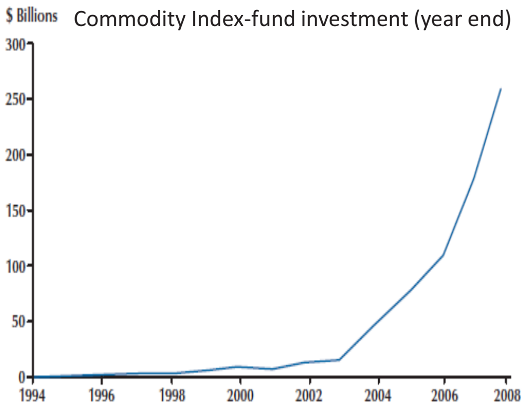
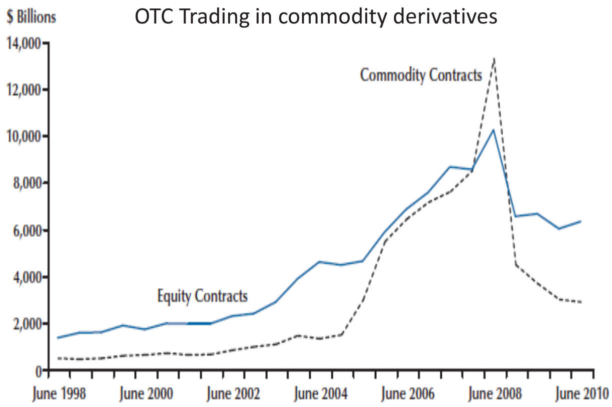
INTRODUCTION

- Optimal behaviors in economic environment have been studied intensively since **Merton (1969, 1971)**
- Standard method of investigation is based on **stochastic control framework** and **Hamilton-Jacobi-Bellman (HJB) equation**, resulting in nonlinear equations which are typically hard to solve
- We introduce the importance of commodities and benefits of commodities within portfolio
- Next, a case of bond-stock-commodity portfolio will be described with a simple interest rate model and mean-reverting commodity prices focusing on individuals with constant relative risk aversion (CRRA) utility
- This results in an explicit investment strategy with hedge variations in interest rates of mixed bond-stock-commodity dynamic portfolio

INTRODUCTION TO COMMODITIES (1)

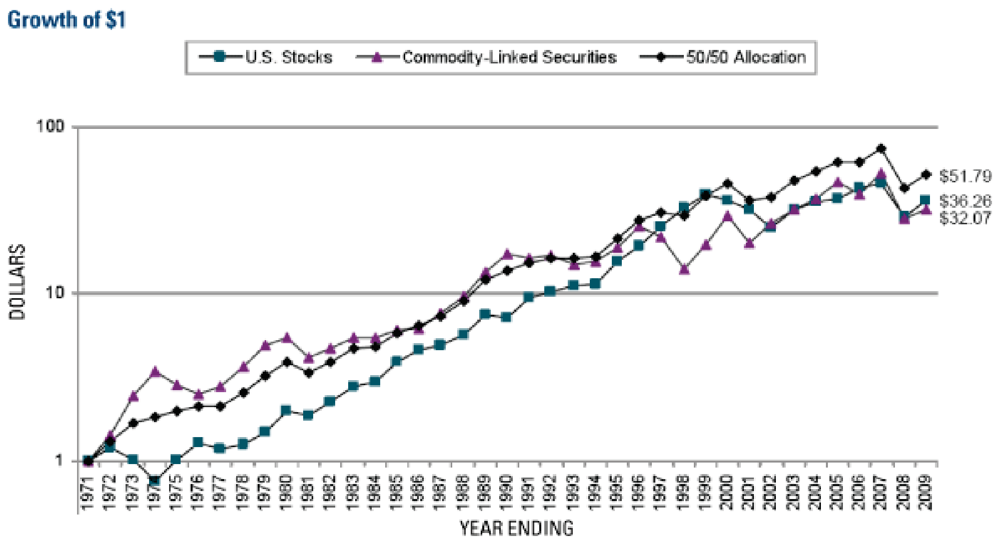
- Commodities have been emerging as an increasingly important class of assets in recent years
- Most existing studies on commodity investment apply the one-period mean-variance optimization framework of Markowitz (1952)
- However, much less research efforts have been devoted to long-term allocation strategy





** Source: Basu and Gavin (2011)*

INTRODUCTION TO COMMODITIES (2)



- An investment of $1 in the S&P 500 in 1971 would by this year have grown to $32.07, and that an identical investment in the S&P GSCI would have increased to $36.26
- However, when we allocate 50/50 in stock and commodity at the start – by 2010, that original dollar would be worth nearly $52

** Source: www.resourceinvestor.com*

INTRODUCTION TO COMMODITIES (3)

	Treasury Bills	TIPS	U.S. Bonds	International Bonds	U.S. Stocks	International Stocks	Commodities	U.S. Inflation
Treasury Bills	1.00	-0.08	0.23	-0.35	0.03	-0.12	-0.10	0.61
TIPS	-0.08	1.00	0.02	0.38	-0.10	-0.04	0.41	0.19
U.S. Bonds	0.23	0.02	1.00	0.14	0.24	-0.03	-0.32	-0.29
International Bonds	-0.35	0.38	0.14	1.00	0.03	0.40	0.15	-0.09
U.S. Stocks	0.03	-0.10	0.24	0.03	1.00	0.58	-0.24	-0.19
International Stocks	-0.12	-0.04	-0.03	0.40	0.58	1.00	-0.07	-0.20
Commodities	-0.10	0.41	-0.32	0.15	-0.24	-0.07	1.00	0.29
U.S. Inflation	0.61	0.19	-0.29	-0.09	-0.19	-0.20	0.29	1.00
Average Correlation (Excluding Inflation)	**-0.06**	**0.1**	**0.05**	**0.12**	**0.09**	**0.12**	**-0.03**	**0.05**

- The table presents the correlation coefficients of annual total returns (1970-2004) which provide intuitive evidence of the low correlation of commodities with traditional asset classes
- Of the 7 asset classes, T-bills and commodities are the only two asset classes with a negative average correlations to the other asset classes

** Source: Ibbotson Associates (2006)*

LITERATURE REVIEW

- **Asset allocation with stochastic interest rates**
  - *Sørensen (1999)* and *Brennan and Xia (2000)*:
    The optimal investment strategy of an investor is a simple combination of the mean-variance optimal portfolio and the zero-coupon bond hedging terms
  - *Munk and Sørensen (2004)*:
    The hedge portfolio is relatively less sensitive to the specific dynamics of interest rates. Thus it is sufficient to apply basic interest rate models when considering problems

- **Asset allocation with commodities**
  - *Dai (2009)*:
    The study suggests that allocation to commodities is needed to optimize the myopic purposes as well as to hedge the stochastic changes of the investment opportunity set in long-term purposes

LITERATURE REVIEW

Financial planner's rules of thumb: Campbell and Viceira (2002)

- Aggressive investors should hold stocks, conservative investors should hold bonds
- Long-term investors can afford to take more stock market risk than short-term investors

INVESTMENT ASSET DYNAMICS (1)

- Assume only one-dimensional state variable, the interest rate, follows an Ito process and particularly consider the case of the Vasicek model

long run mean

$$\mathrm{d}r_t = \kappa(\bar{r} - r_t)\mathrm{d}t - \sigma_r \mathrm{d}Z_{rt}$$

speed of mean reversion

Standard deviation of the short rate

- The process exhibits mean reversion in the sense that if $r_t < \bar{r}$, the short rate is expected to increase over the next period and vice versa
- From Vasicek (1977), the price of a zero-coupon bond with maturity $\bar{T}$ is given by

$$\mathrm{d}B_t^{\bar{T}} = B_t^{\bar{T}}\left[\left(r_t + \sigma_B(r_t,t)\lambda_1\right)\mathrm{d}t + \sigma_B(r_t,t)\mathrm{d}Z_{rt}\right]$$

where $\sigma_B(r_t,t) = \sigma_r b\left(\bar{T} - t\right)$ and $b\left(\bar{T} - t\right) = \frac{1}{\kappa}\left(1 - e^{-\kappa\left(\bar{T}-t\right)}\right)$

INVESTMENT ASSET DYNAMICS (2)

- The price of the single stock is assumed to follow the process

Sharpe ratio of the stock

Volatility of the stock

$$\mathrm{d}S_t = S_t\left[\left(r_t + \sigma_s\psi\right)dt + \rho_{rs}\sigma_s\mathrm{d}Z_{rt} + \sqrt{1-\rho_{rs}^2}\,\sigma_s\mathrm{d}Z_{st}\right]$$

Correlation between bond market returns and stock market returns

- The spot commodity price dynamics is

$$\mathrm{d}C_t = \theta\left(\mu_c - \ln C_t\right)C_t\mathrm{d}t + \sigma_c C_t \mathrm{d}\hat{Z}_{ct}$$

- following Dai (2009), the commodity price process may be adapted to the following process

$$\mathrm{d}V_t = V_t\left[\left(r_t + \sigma_c\lambda_{ct}\right)\mathrm{d}t + \sigma_c\mathrm{d}\hat{Z}_{ct}\right]$$

where $\lambda_{ct} = \lambda_{c1} + \lambda_{c2}X_t,\ \ X_t = \ln C_t$

- With this process, it is possible to set up the dynamics of three assets, bonds, stocks, and commodities consistent with asset allocation problems of Merton (1969, 1971)

INVESTMENT ASSET DYNAMICS (3)

- The dynamics of three assets can be expressed in terms of the following matrix

$$\mathrm{d}\underline{P}_t = \begin{bmatrix} \mathrm{d}B_t^{\bar{T}} \\ \mathrm{d}S_t \\ \mathrm{d}V_t \end{bmatrix} = \mathrm{diag}(P_t)\left[\left(r\underline{1} + \underline{\underline{\sigma}}(r_t, \lambda_{3t}, t)\underline{\lambda}(\lambda_{3t})\right)\mathrm{d}t + \underline{\underline{\sigma}}(r_t, \lambda_{3t}, t)\mathrm{d}\underline{Z}_t\right]$$

where $\underline{\lambda}(\lambda_{3t}) = [\lambda_1 \quad \lambda_2 \quad \lambda_{3t}]^{\mathrm{T}}$

$$\underline{\underline{\sigma}}(r_t, \lambda_{3t}, t) = \begin{bmatrix} \sigma_B(r_t, t) & 0 & 0 \\ \rho_{rs} & \bar{\rho}\sigma_s & 0 \\ \rho_{cr} & \rho_{sc} & \tilde{\rho}\sigma_s \end{bmatrix}, \quad \bar{\rho} = \sqrt{1-\rho_{rs}^2}, \quad \tilde{\rho} = \sqrt{1-\rho_{cr}^2-\rho_{sc}^2}$$

- The market price of risk, λ_{3t}, can be expressed as

$$\mathrm{d}\lambda_{3t} = \theta\left(\bar{\lambda}_{3t} - \lambda_{3t}\right)\mathrm{d}t + \left(\lambda_2\sigma_c/\tilde{\rho}\right)[\rho_{cr} \quad \rho_{sc} \quad \tilde{\rho}]\mathrm{d}\underline{Z}_t$$

where $\bar{\lambda}_{3t} = (1/\tilde{\rho})\left(\lambda_{c2}\alpha + \lambda_{c1} - \rho_{cr}\lambda_1 - \rho_{sc}\lambda_2\right)$

OPTIMAL ASSET ALLOCATION (1)

- Wealth process can be written as

$$\mathrm{d}W_t = W_t\left[r_t + \underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}(r_t, \lambda_{3t}, t)\underline{\lambda}(\lambda_{3t})\right]\mathrm{d}t + W\underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}(r_t, \lambda_{3t}, t)\mathrm{d}\underline{Z}_t$$

Investment strategy

- An investor is assumed to maximize utility from the terminal wealth with respect to a power utility function

$$J\left(W, r, \lambda_{3t}, t\right) = \sup E_{W,r,\lambda_{3t},t}\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]$$

- The Hamilton-Jacobi-Bellman (HJB) equation of the problem is

$$0 = WJ_W\underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}(r_t, \lambda_{3t}, t)\underline{\lambda}(\lambda_{3t}) + \frac{1}{2}J_{WW}W^2\underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}(r_t, \lambda_{3t}, t)\underline{\underline{\sigma}}(r_t, \lambda_{3t}, t)^{\mathrm{T}}\underline{\pi}_t + J_t$$

$$+ W\underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}(r_t, \lambda_{3t}, t)\underline{\underline{v}}(r, \lambda_{3t})J_{W\underline{x}} + rWJ_W + J_{\underline{x}}^{\mathrm{T}}\underline{m}(r, \lambda_{3t}) + \frac{1}{2}\mathrm{tr}\left(J_{\underline{xx}}\underline{\underline{\Sigma}}(r, \lambda_{3t})\right)$$

where $\underline{\underline{\Sigma}}(r, \lambda_{3t}) = \underline{\underline{v}}(r, \lambda_{3t})^{\mathrm{T}}\underline{\underline{v}}(r, \lambda_{3t})$

OPTIMAL ASSET ALLOCATION (2)

- **Theorem 1:** Solving the equation, we obtain the following strategy

$$\pi_B = \frac{1}{\gamma\sigma_B(r,t)}\left[\lambda_1 - \frac{\lambda_2\rho_{rs}}{\bar{\rho}} - \frac{\lambda_{3t}}{\bar{\rho}\tilde{\rho}}\left(\bar{\rho}\rho_{cr} - \rho_{rs}\rho_{sc}\right)\right] + \frac{\gamma-1}{\gamma}\frac{\sigma_r A_1(\tau)}{\sigma_B(r,t)}$$

$$\pi_S = \frac{1}{\gamma\sigma_S}\left(\frac{\lambda_2}{\bar{\rho}} - \frac{\lambda_{3t}\rho_{sc}}{\bar{\rho}\tilde{\rho}}\right)$$

$$\pi_V = \frac{\lambda_{3t}}{\gamma\tilde{\rho}\sigma_C} - \frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\left[A_2(\tau) + A_3(\tau)\lambda_{3t}\right]$$

Speculative terms | Hedging terms

- As previous literatures suggest, investment allocation consists of two terms, speculative and hedging terms
- Stock allocation is independent from the investment horizon, contradicting with traditional advice that the stock weight should increase with investment horizon

ALLOCATION ANALYSIS (1)

Zero-coupon bond allocation

$$\pi_B = \frac{1}{\gamma\sigma_B(r,t)}\left[\lambda_1 - \frac{\lambda_2\rho_{rs}}{\bar{\rho}} - \frac{\lambda_{3t}}{\bar{\rho}\tilde{\rho}}\left(\bar{\rho}\rho_{cr} - \rho_{rs}\rho_{sc}\right)\right] + \frac{\gamma-1}{\gamma}\frac{\sigma_r A_1(\tau)}{\sigma_B(r,t)}$$

positive or negative ?

- The term λ_{3t} is assumed to be negatively correlated to the spot commodity price as in Schwartz (1997) and Dai (2009)
- Signs and magnitudes of correlation parameters will help ascertain the direction of bond allocation caused by a rise in commodity price
- If the allocation in bonds is the zero-coupon maturing at the end of the investment horizon, the hedge term will not depend on investment time horizon
- Therefore, it is clear that the hedge position of a more risk-averse investor is larger than that of a less risk-averse investor

ALLOCATION ANALYSIS (2)

Stock allocation

$$\pi_S = \frac{1}{\gamma\sigma_S}\left(\frac{\lambda_2}{\bar{\rho}} - \frac{\lambda_{3t}\rho_{sc}}{\bar{\rho}\tilde{\rho}}\right)$$

- if λ_{3t} is negatively correlated to the commodity price, the stock investment has a negative relationship with the commodity price
- when the commodity price reduces, this suggests in the increase in the stock price; and investors, therefore, should invest in stock

Commodity allocation

$$\pi_V = \frac{\lambda_{3t}}{\gamma\tilde{\rho}\sigma_C} - \frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\left[A_2(\tau) + A_3(\tau)\lambda_{3t}\right]$$

- Due to the positivity of the term $A_3(\tau)$, there is an inverse relationship between the commodity price and the hedge portfolio
- An investor should invest in commodities when the commodity price decreases and reduce the portion when price increases

ALLOCATION ANALYSIS (3)

Horizon effect

- By differentiating the commodity wealth fraction with respect to time horizon, we obtain

$$\frac{\partial\pi_V}{\partial\tau} = -\frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\left[A_2'(\tau) + A_3'(\tau)\lambda_{3t}\right]$$

- It can be shown that $A_3'(\tau)$ is always positive while $A_2'(\tau)$ can be either positive or negative depending on parameters
- Thus, it is inconclusive whether the investment horizon has a positive or negative effect to commodity investment

WELFARE ANALYSIS (1)

Welfare analysis

- The loss is assumed as the percentage L of extra initial wealth that is necessary to bring the investor to the same utility level as the investor considering investment in a commodity

$$J^{NC}\left(W(1+L), r, \lambda_{3t}, t\right) = J\left(W, r, \lambda_{3t}, t\right) = \frac{1}{1-\gamma}\left(W e^{H(r,\lambda_{3t},\tau)}\right)^{1-\gamma}$$

- We obtain

$$L = \exp\left\{H(r, \lambda_{3t}, \tau) - H^{NC}(r, \tau)\right\} - 1$$

WELFARE ANALYSIS (2)

- Inserting both function *H* into the wealth loss function, we obtain

$$L = \exp\left\{A_2(\tau)\lambda_{3t} + \frac{1}{2}A_3(\tau)\lambda_{3t}^2 + \left(\theta\bar{\lambda}_{3t} - \frac{\gamma-1}{\gamma}\right)\int_0^\tau A_1(s)\mathrm{d}s + \frac{1}{2}\sigma_*^2\int_0^\tau A_3(s)\mathrm{d}s \right.$$
$$\left. + \frac{\gamma-1}{\gamma}\frac{\lambda_{c2}}{\tilde{\rho}}\sigma_c\sigma_r\rho_{cr}\int_0^\tau A_1(s)A_2(s)\mathrm{d}s + \sigma_*^2\int_0^\tau A_2^2(s)\mathrm{d}s\right\} - 1$$

- With very high or very low commodity prices, the square of market price of risk will be high, leading to an increase in the welfare loss

- On the other hand, the standard level of commodity values can reduce wealth loss. This may be reckoned as the opportunity loss from extreme movement in commodity prices

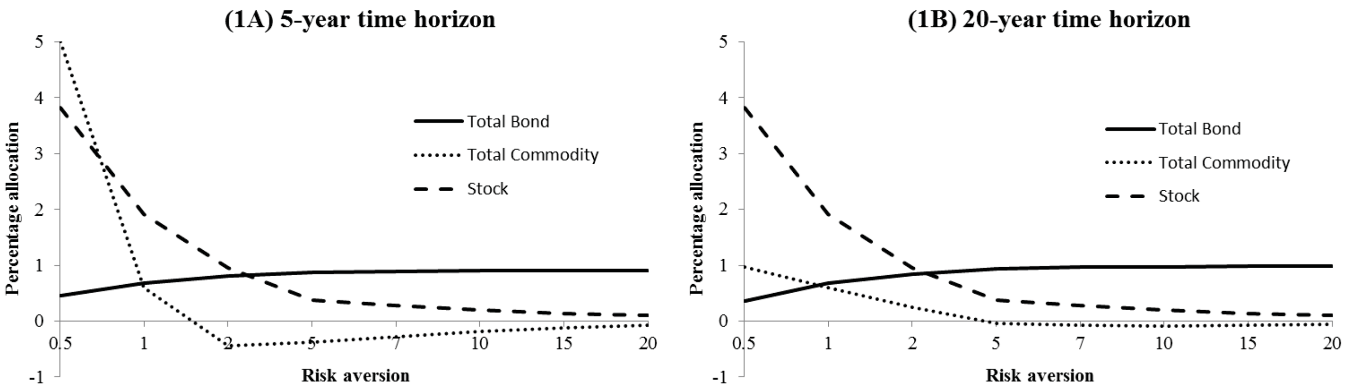
A NUMERICAL EXAMPLE (1)

Notes on parameters

This section illustrates the strategy in Theorem 1 with historical estimates of mean returns, standard deviations, and correlations as representative of future investment opportunities. Estimates are taken from Munk (2010), Dai (2009), and Ibbotson Associates (2006). The average real return on the U.S. stock market is 8.70% with a standard deviation of σ_S = 20.20%, while the average real return on bonds is 2.10% with a standard deviation of 10.00%. The average real U.S. short-term interest rate is $\overline{r}$ = 1.00%. and the correlation between stock returns and bond returns is ρ_{rs} = 0.20. The average long-run mean of the commodity price, μ_c, is 5.02, the degree of mean-reversion, θ, is 0.12, with a standard deviation of σ_c = 19.90%. Market prices of risk, λ_{c1} and λ_{c2} equal 3.32 and -0.67. Correlation coefficients, ρ_{rs}, ρ_{sc}, and ρ_{cr} are 0.20, -0.10, and -0.30, respectively. Finally, the initial spot commodity price is set at 130, the same as in Dai (2009).

A NUMERICAL EXAMPLE (1)

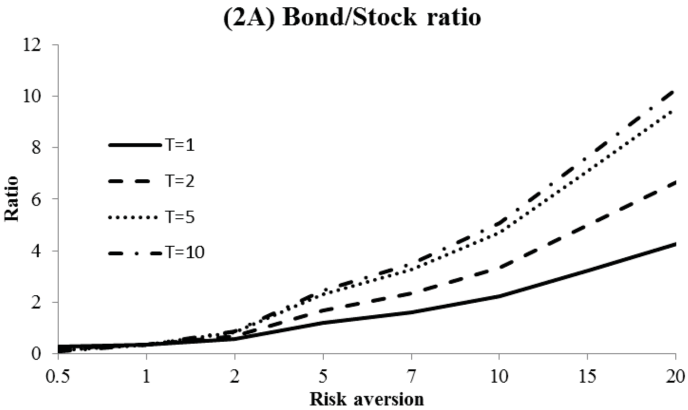
Total portfolio with risk aversion

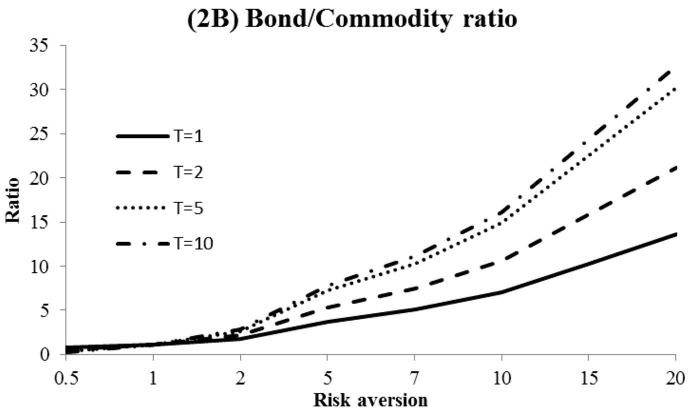


- Investment in bonds and risk aversion factor has a positive relationship implying that more conservative investors take larger positions in bonds
- Conservative investors allocate less in stocks compared to aggressive investors
- The relation between risk aversion factor and commodity investment is indeterminate

A NUMERICAL EXAMPLE (2)

Bond/stock and bond/commodity ratio

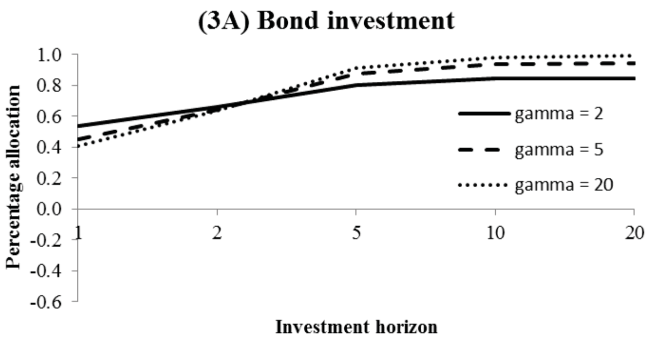


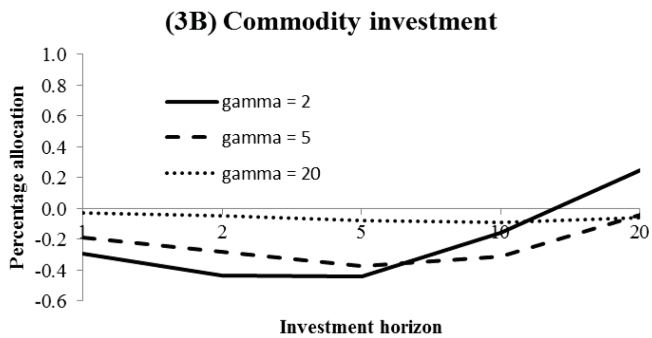


- For each investment horizon, bond/stock and bond/commodity ratios increases with increasing risk aversion
- Investors with risk aversion less than one, the ratios decrease with increasing time horizon
- For risk aversion factors greater than one, the ratios increase with increasing time horizon, contradicting the professional advice

A NUMERICAL EXAMPLE (3)

Horizon effect

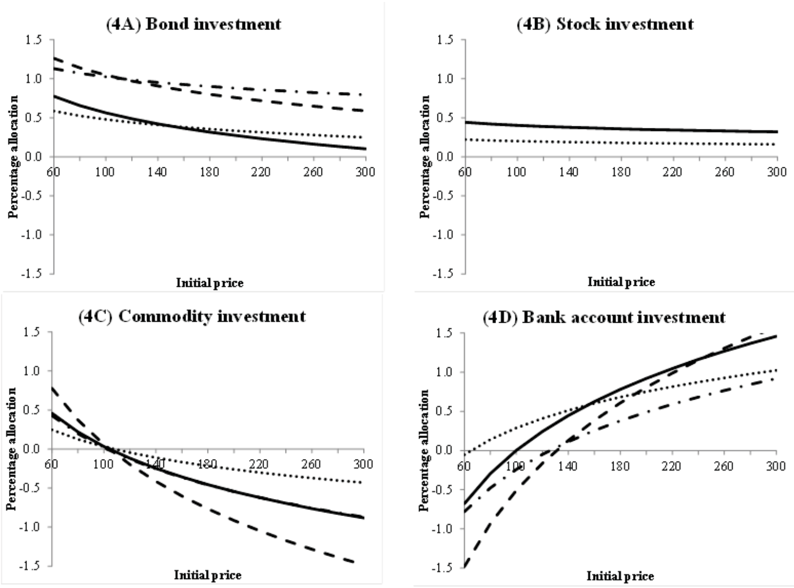




- Bond investment has a positive relationship with the time horizon
- For medium-term commodity investment, investments decrease for longer investment horizons. However, the opposite applies for long-term investment
- Additionally, conservative investors slowly increase their portions in commodity allocation compared to aggressive investors

A NUMERICAL EXAMPLE (4)

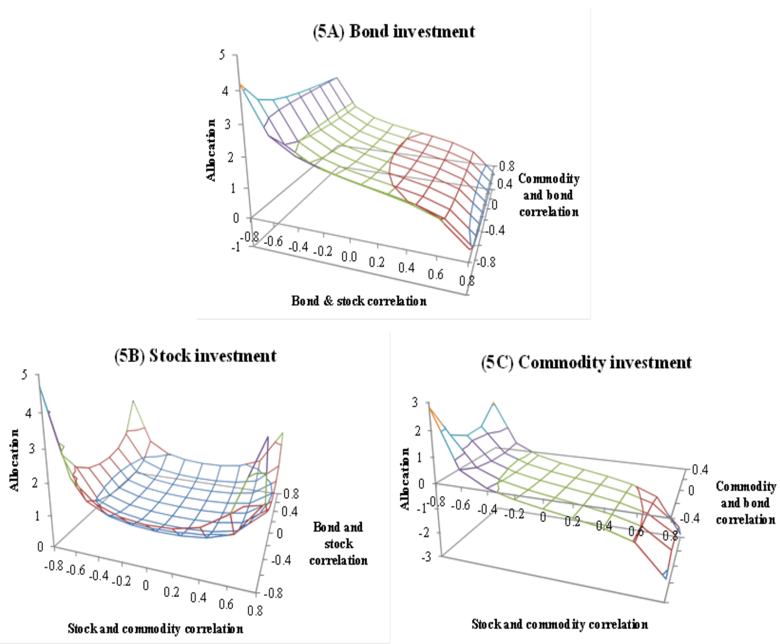
Change in spot initial price



- Commodity price increase tends to decrease investors' positions in risky assets
- Moreover, conservative investors smoothly reduce their positions compared to aggressive investors

A NUMERICAL EXAMPLE (5)

Correlation effect



- If relevant correlations convert to -1, investors tend to increase their portions in that asset
- Correlation between stock returns and commodity returns reduces with increasing investment in commodity
- For stock investment, surprisingly, when absolute values of relevant correlations convert to 1, investors increase their positions in stock allocation

SUMMARY

- We set up investment asset dynamics and optimal asset allocation using the dynamic programming methodology, resulting in explicit investment strategy with hedge variations in interest rates of mixed bond-stock-commodity portfolio
- Positions in stocks and commodities have negative relationships with investor's risk tolerance, while positions in bonds have the opposite result
- Bond investment has a positive relationship with the time horizon. A relationship between investment in commodity and time horizon, however, is more complicated
- All positions in risky assets decrease with increasing commodity prices and thus position in bank account increases to balance the total portfolio
- The study of correlation and financial allocation results in various outcomes depending on estimated parameters. In summary, allocation in each asset will be increased when relevant correlations convert to -1.

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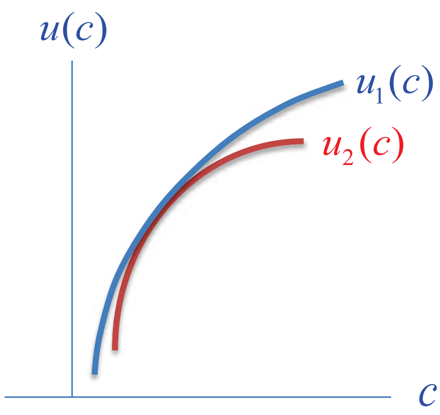
APPENDIX: COEFFICIENT OF RISK AVERSION

- Normally, the higher the curvature of utility function $u(x)$, the higher the risk aversion
- Arrow and Pratt derived what are known as the coefficient of **absolute risk aversion (ARA)** and the coefficient of **relative risk aversion (RRA)**, to characterize degrees of risk aversion



$$r_A(c,u) := -\frac{\partial^2 u / \partial c^2}{\partial u / \partial c} = \frac{\gamma}{c} \quad \text{and} \quad r_R(c,u) := -\frac{c\partial^2 u / \partial c^2}{\partial c / \partial c} = c \cdot r_A(c,u) = \gamma$$

- For standard risk averse investor, the coefficient of RRA is expected to be **non-increasing**
- In particular, the individual becomes **less risk averse** with regard to gambles that are proportional to his wealth **as his wealth increase**